Algebra-Aufgaben: Komplexe Zahlen 2

① a) $z^2 - 4z + 13 = 0 \quad \Rightarrow \quad z_{1,2} = \dfrac{-(-4) \pm \sqrt{(-4)^2 - 4 \cdot 1 \cdot 13}}{2 \cdot 1} = \dfrac{4 \pm \sqrt{-36}}{2} = 2 \pm \sqrt{-9} = 2 \pm 3i$

b) $81z^2 + 25 = 0 \quad \Leftrightarrow \quad z = \dfrac{-25}{81} \quad \Rightarrow \quad z_{1,2} = \pm\sqrt{\dfrac{-25}{81}} = \pm \dfrac{5}{9} \cdot i$

c) $z^2 + z = -1 \quad \Leftrightarrow \quad z^2 + z + 1 = 0 \quad \Rightarrow \quad z_{1,2} = \dfrac{-1 \pm \sqrt{1-4}}{2} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} \cdot i$

② a) $(7-5i) - (2+i) - (-5+2i) = 10 - 8i$

b) $(7-5i)\left[(2+i) \cdot (-5+2i)\right] = (7-5i) \cdot (-10 + 4i - 5i + 2i^2)$

$= (7-5i) \cdot (-12 - i)$

$= -84 - 7i + 60i + 5i^2 = -89 + 53i$

c) $Re(z_1 + 4z_2) = Re\left((7-5i) + 4 \cdot (2+i)\right)$

$= Re(15 \pm \ldots) = 15$

d) $Im(z_2^2 \cdot z_4) = Im\left((2+i)^2 \cdot (-10-3i)\right)$

$= Im\left((3+4i)(-10-3i)\right)$

$= Im(\ldots - 9i - 40i \pm \ldots) = -49$

e) $\overline{z_2} = \overline{2+i} = 2 - i$

f) $\overline{z_3^2} = \overline{(-5+2i)^2} = \overline{25 - 20i + 4i^2} = 21 + 20i$

g) $z_4^{-1} = \dfrac{1}{-10-3i} = \dfrac{1 \cdot (-10+3i)}{(-10-3i)(-10+3i)} = \dfrac{-10+3i}{100 - 9i^2} = -\frac{10}{109} + \frac{3}{109} \cdot i$

$(-0{,}092 + 0{,}028i)$

h) $\dfrac{Re(z_1)}{Re(z_2)} = \dfrac{7}{2}$

③ a) $\dfrac{(5+5i)-\overline{(5+5i)}}{(1+2i)\cdot\overline{(1+2i)}} = \dfrac{(5+5i)-(5-5i)}{(1+2i)(1-2i)}$

$= \dfrac{10i}{1-4i^2} = \underline{\underline{2i}}$

b) $\dfrac{4+\sqrt{2}i}{\sqrt{2}-4i} = \dfrac{(4+\sqrt{2}i)(\sqrt{2}+4i)}{(\sqrt{2}-4i)(\sqrt{2}+4i)}$

$= \dfrac{4\sqrt{2}+16i+2i+4\sqrt{2}i^2}{2-16i^2} = \dfrac{18i}{18} = \underline{\underline{i}}$

④ a) $5z = 8iz + (81-5i) \iff 5z-8iz = 81-5i$

$\iff (5-8i)z = 81-5i$

$\iff \underline{\underline{z}} = \dfrac{(81-5i)\cdot(5+8i)}{(5-8i)(5+8i)} = \dfrac{405+648i-25i-40i^2}{25-64i^2}$

$= \dfrac{445+623i}{85} = \underline{\underline{5+7i}}$

b) $\dfrac{z-3i-3}{z+2+4i} = i \iff z-3i-3 = zi+2i+4i^2$

$\iff z(1-i) = 5i-1$

$\iff \underline{\underline{z}} = \dfrac{5i-1}{1-i} = \dfrac{(5i-1)(1+i)}{(1-i)(1+i)} = \dfrac{5i+5i^2-1-i}{1-i^2} = \dfrac{-6+4i}{2} = \underline{\underline{-3+2i}}$

⑤ Sei gewählt $z = x+iy$

a) $\left.\begin{aligned} \bar{z} = z &\iff x-iy = x+iy \\ &\iff -iy = iy \\ &\iff y = 0 \end{aligned}\right\} \Rightarrow \underline{\underline{\bar{z} = z \iff z \in \mathbb{R}}}$

b) $Re(z) = Re(\bar{z}) \iff x = x \Rightarrow \underline{\underline{Re(z) = Re(\bar{z}) \iff z \in \mathbb{C}}}$

c) $\left.\begin{aligned} Im(z) + Im(-z) &= Im(x+iy) + Im(-x-iy) \\ &= y + (-y) = 0 \end{aligned}\right\} \Rightarrow \underline{\underline{Im(z) + Im(-z) = 0 \iff z \in \mathbb{C}}}$

⑥ Beh. $z\bar{z} = |z|$, $z \in \mathbb{C}$, $z = x+iy$

Beweis. $\left.\begin{aligned} z\cdot\bar{z} &= (x+iy)(x-iy) = x^2-i^2y^2 = x^2+y^2 \\ |z| &= \sqrt{x^2+y^2} \end{aligned}\right| \neq \Rightarrow \underline{\underline{\text{Beh. ist falsch!}}}$

(7) Beh.: Mit z ist auch $\bar{z}$ "Lösung".

Beweis: Algebraische Gl. n-ten Grades mit reellen Koeffizienten

$$a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \dots + a_1 x + a_0 = 0 \quad , \text{ mit } a_i \in \mathbb{R},\ i \in \{0,1,\dots,n\},\ a_n \neq 0$$

z ist Lösg.:

$$a_n z^n + a_{n-1} z^{n-1} + a_{n-2} z^{n-2} + \dots + a_1 z + a_0 = 0$$

$$\Leftrightarrow \quad \overline{a_n z^n + a_{n-1} z^{n-1} + a_{n-2} z^{n-2} + \dots + a_1 z + a_0} = \bar{0}$$

$$\overset{6b)}{\Leftrightarrow} \quad \overline{a_n z^n} + \overline{a_{n-1} z^{n-1}} + \overline{a_{n-2} z^{n-2}} + \dots + \overline{a_1 z} + \overline{a_0} = 0$$

$$\overset{6c)}{\Leftrightarrow} \quad \bar{a}_n \bar{z}^n + \bar{a}_{n-1} \bar{z}^{n-1} + \dots + \bar{a}_1 \bar{z} + \bar{a}_0 = 0$$

da reell Koeff.

$$\Leftrightarrow \quad \underline{a_n \bar{z}^n + a_{n-1} \bar{z}^{n-1} + \dots + a_1 \bar{z} + a_0 = 0} \quad \Rightarrow \bar{z} \text{ ist Lösg}$$

8 c) $\underline{x^3 - 12x^2 + cx + d = 0 \quad \text{mit} \quad x_1 = 3+i \text{ als Lösg.}}$

$$\Rightarrow x_1^3 - 12x_1^2 + cx_1 + d = (3+i)^3 - 12\cdot(3+i)^2 + c(3+i) + d = 0$$

$$\Leftrightarrow 27 + 27i + 9i^2 + i^3 - 12(9 + 6i + i^2) + 3c + ci + d = 0$$

$$\Leftrightarrow 27 - 9 - 108 + 12 + 27i - i - 72i + 3c + d + ci = 0$$

$$\Leftrightarrow -78 - 46i + 3c + d + ci = 0$$

$$\Leftrightarrow ci = 46i \wedge 78 = 3c + d$$

$$\Leftrightarrow \underline{\underline{c = 46 \wedge d = -60}}$$

$\underline{\underline{x_2 = 3 - i}}$

Bem. $(x - z)(x - \bar{z}) = x^2 - x(z + \bar{z}) + z\bar{z}$

$= x - 2x\cdot Re(z) + Re(z)^2 + Im(z)^2$

$$\Rightarrow (x - x_1)(x - x_2) = \underline{x^2 - 6x + 10}$$

$$\Rightarrow (x^3 - 12x^2 + 46x - 60) : (x^2 - 6x + 10) = \underline{x - 6} \quad \Rightarrow \quad \underline{\underline{x_3 = 6}}$$

$-(x^3 - 6x^2 + 10x)$

$-6x^2 + 36x - 60$

$(-6x^2 + 36x - 60)$

$-$

5) $x^4 - 50x^2 + ax + b = 0$ mit $x_1 = 1+i$ ist Lösg.

$$x_1^4 - 50x_1^2 + a \cdot x_1 + b = (1+i)^4 - 50(1+i)^2 + a\cdot(1+i) + b \stackrel{!}{=} 0$$

$$\Leftrightarrow 1 + 4i + 6i^2 + 4i^3 + i^4) - 50(1 + 2i + i^2) + a + ai + b = 0$$

$$\Leftrightarrow 1 + 4i - 6 - 4i + 1 \quad - 50 - 100i + 50 + a + ai + b = 0$$

$$\Leftrightarrow -4 + a + b \quad - 100i + ai = 0$$

$$\Leftrightarrow \underline{\underline{a = 100}}, \ \underline{\underline{b = -96}}$$

$\underline{\underline{x_2 = 1-i}}$

$$(x - x_1)(x - x_2) = \underline{x^2 - 2x + 2}$$

$$\Rightarrow (x^4 - 50x^2 + 100x - 96) : (x^2 - 2x + 2) = x^2 + 2x - 48 = \underline{(x-6)(x+8)}$$

$$\underline{-(x^4 - 2x^3 + 2x^2)}$$

$$2x^3 - 52x^2 + 100x - 96$$

$$\underline{-(2x^3 - 4x^2 + 4x)}$$

$$-48x^2 + 96x - 96$$

$$\underline{-(-48x^2 + 96x - 96)}$$

$$-$$

$$\Rightarrow \underline{\underline{x_3 = 6, \ x_4 = -8}}$$

c) $\underline{x^3 + ax = 100}$, $\underline{\text{mit } x_1 = -2+iy \text{ als Lösg.}}$

$$x_1^3 + a\cdot x_1 = (-2+iy)^3 + a\cdot(-2+iy) \overset{!}{=} 100$$

$$\Leftrightarrow -8 + 12iy + 6y^2 - iy^3 - 2a + ayi \overset{!}{=} 100$$

$$\Leftrightarrow 6y^2 - 8 - 2a = 100 \quad \wedge \quad (12y - y^3 + ay)i = 0\cdot i$$

$$\Leftrightarrow \underline{6y^2 = 108 + 2a} \quad \wedge \quad y(12 + a - y^2) = 0$$

$$\Leftrightarrow \underline{y^2 = 12 + a}$$

$$\Rightarrow 108 + 2a = 6\cdot(12+a)$$

$$\Leftrightarrow \underline{\underline{a = 5}} \qquad \Rightarrow y = \sqrt{21}$$

$$\Rightarrow \underline{\underline{x_1 = -2+\sqrt{21}}}, \quad \underline{\underline{x_2 = -2-\sqrt{21}}}$$

$$\Rightarrow (x^3 + 9x - 100) : \overbrace{(x^2 + 4x + 25)}^{=(x-x_1)(x-x_2)} = \underline{x-4} \qquad \Rightarrow \underline{\underline{x_3 = 4}}$$

$$\begin{array}{l} \underline{-(x^3 + 4x^2 + 25x)} \\ \quad -4x^2 - 16x - 100 \\ \underline{-(-4x^2 - 16x - 100)} \\ \quad\quad - \end{array}$$

d) $\underline{x^4 - 12x^3 + cx^2 + bx + 72 = 0}$, mit zwei rein-imaginären Lsg. $\Rightarrow x_{1,2} = \pm iy$

eine reelle Lsg mit der Vielfachheit $= 2 \Rightarrow x_{3,4} = r$

$$\cdot \; x_1^4 - 12x_1^3 + cx_1^2 + bx_1 + 72 = i^4y^4 - 12\cdot i^2 iy^3 + c\cdot i^2y^2 + biy + 72 \overset{!}{=} 0$$

$$\Leftrightarrow y^4 + 12y^3 i - cy^2 + byi + 72 = 0$$

$$\Leftrightarrow \underline{y^4 - cy^2 - 72 = 0} \;\wedge\; \underline{(12y^3 + by)\cdot i = 0i}$$

$$\cdot \; x_2^4 - 12x_2^3 + cx_2^2 + bx_2 + 72 = (-i)^4y^4 - 12\cdot(-i)^2\cdot(-i)y^3 + c\cdot(-iy)^2 + b\cdot(-iy) + 72 = 0$$

$$\Leftrightarrow y^4 - 12y^3 i - cy^2 - byi + 72 = 0$$

$$\Leftrightarrow \underline{y^4 - cy^2 + 72 = 0} \;\wedge\; \underline{(12y^3 + by)i = 0i}$$

$y_1 = 0$

$y_{2,3} = \pm\sqrt{b/12}$

$y^2 = \tilde{y}$

$$\tilde{y}_{1,2} = \frac{c \pm \sqrt{c^2 - 4\cdot 72}}{2}$$

⋮ bringt nichts!...

$$x^4 - 12x^3 + cx^2 + bx + 72 = (x+iy)(x-iy)(x-r)^2$$

$$= (x^2 - (i^2y^2))(x-r)^2$$

$$= (x^2 + y^2)(x^2 - 2xr + r^2)$$

$$= x^4 - 2x^3r + x^2r^2 + x^2y^2 - 2xy^2r + y^2r^2$$

$$= x^4 - 2x^3r + (r^2 + y^2)x^2 - 2ry^2x + y^2r^2$$

$$\Leftrightarrow -12x^3 = -2x^3r \qquad \Rightarrow \underline{\underline{r = 6}} = x_{3,4}$$

und

$$\Leftrightarrow cx^2 = (r^2 + y^2)x^2 \qquad \Rightarrow \underline{\underline{c = 38}}$$

und

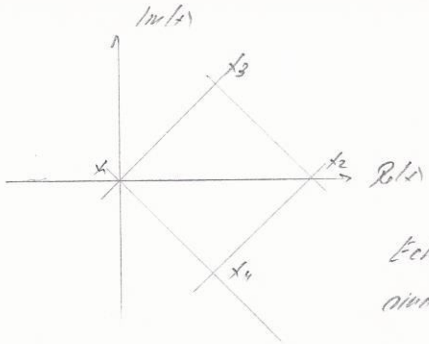
$$\Leftrightarrow bx = -2ry^2x \qquad \Rightarrow \underline{\underline{b = -24}}$$

und

$$\Leftrightarrow 72 = y^2r^2 \qquad \Rightarrow y_{1,2} = \pm\sqrt{2}$$

$$\Rightarrow \underline{\underline{x_{1,2} = \pm i\sqrt{2}}}$$

e) $\underline{x^4 - 4x^3 + ax^2 + bx + c = 0}$



Ecken des Quadrats sind Lösg., $x_1 = (0|0)$

$\underline{\underline{x_1 = 0}} \Rightarrow \underline{\underline{c = 0}}$

$\underline{x_{3,4} = r \pm is}$

$\underline{x_2 = 2r}$

$\Rightarrow x^4 - 4x^3 + ax^2 + bx = 0$

$\Leftrightarrow \underbrace{x(x^3 - 4x^2 + ax + b)}= 0$

$= (x - x_2) \cdot \underbrace{(x - x_3)(x - x_4)}$

$= (x - 2r) \cdot \underline{(x^2 - 2x \cdot r + (r^2 + s^2))}$ (da $x_3 = \bar{x}_4$)

$= x^3 - 2x^2 r + x(r^2 + s^2) - 2x^2 r + 4xr^2 - 2r^3 - 2rs^2$

$= x^3 - 4x^2 r + (4r^2 + r^2 + s^2)x - 2r^3 - 2rs^2$

Koeff. vergleich $\Leftrightarrow \underline{-4 = -4r} \wedge a = 5r^2 + s^2 \wedge 2r^3 + 2rs^2 = -b$

$\Leftrightarrow \underline{r = 1} \Rightarrow \underline{\underline{x_2 = 2}}$

$\Rightarrow \underline{x_{3,4} = 1 \pm is}$, $s = 1$ ↑ Quadrat

$\Rightarrow \underline{\underline{x_{3,4} = 1 \pm i}}$

$\Rightarrow \underline{\underline{a}} = 5 \cdot 1^2 + 1 = \underline{\underline{6}}$

$\Rightarrow \underline{\underline{b}} = -(2 \cdot 1^3 + 2 \cdot 1 \cdot 1) = \underline{\underline{-4}}$