Algebra-Aufgaben: Komplexe Zahlen 4

(1) "trigo → kart.":

a) $2\,\mathrm{cis}\,\pi = 2\cdot(-1+i\cdot 0) = \underline{-2}$

b) $6\cdot\mathrm{cis}\,\frac{5\pi}{3} = 6\cdot\left(\frac{1}{2} - i\cdot\frac{\sqrt{3}}{2}\right) = \underline{3 - i\cdot 3\sqrt{3}}$

c) $2\left(\cos\frac{7\pi}{6} + i\cdot\sin\frac{7\pi}{6}\right) = 2\cdot\left(-\frac{\sqrt{3}}{2} - i\cdot\frac{1}{2}\right) = \underline{-\sqrt{3} - i}$

d) $4\cdot\left(\cos\frac{\pi}{2} + i\cdot\sin\frac{\pi}{2}\right) = 4\cdot(0 + i\cdot 1) = \underline{4i}$

(2) "kart. → trigo":

a) $2 + i\pi \overset{\text{1. Quadr.}}{\Rightarrow} \left.\begin{aligned} r &= \sqrt{2^2+\pi^2} = 3{,}724 \\ \varphi &= \tan^{-1}\left(\frac{\pi}{2}\right) = 57{,}518^\circ \end{aligned}\right\} \Rightarrow \underline{3{,}724\cdot\mathrm{cis}\,57{,}518^\circ}$

b) $-1 + i \overset{\text{2. Quadr.}}{\Rightarrow} \left.\begin{aligned} r &= \sqrt{2} \\ \varphi &= 180^\circ + \tan^{-1}\left(\frac{1}{-1}\right) = 135^\circ \end{aligned}\right\} \Rightarrow \underline{\sqrt{2}\cdot\mathrm{cis}\,135^\circ}$

c) $-4i = \underline{4\cdot\mathrm{cis}\,270^\circ}$ d) $2 = \underline{2\cdot\mathrm{cis}\,0^\circ}$

e) $3 - i \overset{\text{4. Quadr.}}{\Rightarrow} \left.\begin{aligned} r &= \sqrt{10} \\ \varphi &= 360^\circ + \tan^{-1}\left(\frac{-1}{3}\right) \end{aligned}\right\} \Rightarrow \underline{\sqrt{10}\cdot\mathrm{cis}\,341{,}565^\circ}$

f) $-1 - i \overset{\text{3. Quadr.}}{\Rightarrow} \left.\begin{aligned} r &= \sqrt{2} \\ \varphi &= 180^\circ + \tan^{-1}\left(\frac{-1}{-1}\right) = 225^\circ \end{aligned}\right\} \Rightarrow \underline{\sqrt{2}\cdot\mathrm{cis}\,225^\circ}$

(3) a)
$$\begin{aligned}(\cos 15^\circ + i\cdot\sin 15^\circ)\cdot(\cos 60^\circ + i\cdot\sin 60^\circ) &= \underline{\mathrm{cis}\,75^\circ} \\ &= 1\cdot(0{,}259 + i\cdot 0{,}966) \\ &= \underline{0{,}259 + i\cdot 0{,}966}\end{aligned}$$

b)
$$\begin{aligned}\mathrm{cis}\,\frac{\pi}{6} : \mathrm{cis}\,\frac{\pi}{3} = \mathrm{cis}\left(-\frac{\pi}{6}\right) &= \underline{\mathrm{cis}\,\frac{11\pi}{6}} \\ &= 1\cdot\left(0{,}866 + i\cdot\left(-\tfrac{1}{2}\right)\right) \\ &= \underline{\tfrac{\sqrt{3}}{2} - \tfrac{1}{2}i}\end{aligned}$$

c) $(1+i) + \sqrt{2}\cdot \text{cis}\, 135° = (1+i) + \sqrt{2}\cdot\left(-\frac{\sqrt{2}}{2} + i\cdot\frac{\sqrt{2}}{2}\right)$

$= (1+i) + (-1+i) = \underline{\underline{2i}} = \underline{\underline{2\cdot \text{cis}\, 90°}}$

d) $(1+i)\cdot\sqrt{2}\cdot \text{cis}\, 135° = (1+i)\cdot(-1+i) = i^2 - 1^2 = \underline{\underline{-2}} = \underline{\underline{2\cdot \text{cis}\, 180°}}$

" oder

$\sqrt{2}\cdot \text{cis}\, 45° \cdot \sqrt{2}\cdot \text{cis}\, 135° = \sqrt{2}\cdot\sqrt{2}\cdot \text{cis}\,(45°+135°) = \underline{\underline{2\cdot \text{cis}\, 180°}}$

e) $(1+\sqrt{3}\cdot i) + 4\cdot \text{cis}\, 120° = (1+\sqrt{3}\cdot i) + 4\cdot\left(-\tfrac{1}{2} + i\cdot\frac{\sqrt{3}}{2}\right) = \underline{\underline{-1+3\sqrt{3}\cdot i}}$

$\overset{\text{2. Quadr.}}{=} \underline{\underline{\sqrt{28}\cdot \text{cis}\, 100{,}893}}$

f) $(\text{cis}\, 30° : \text{cis}\, 60°) : \text{cis}\, 200° = \text{cis}(-30°) : \text{cis}\, 200$

$= \text{cis}(-230°) = \underline{\underline{\text{cis}\, 130°}}$

$= \underline{\underline{-0{,}643 + i\cdot 0{,}766}}$

g) $\dfrac{\cos 210° - i\cdot \sin 210°}{\cos 150° + i\cdot \sin 150°} = \dfrac{\cos(-210°) + i\cdot\sin(-210°)}{\cos 150° + i\cdot \sin 150°}$

$= \dfrac{\text{cis}(-210°)}{\text{cis}(150°)} = \text{cis}\,(-210° - 150°) = \text{cis}\,(-360°) = \underline{\underline{\text{cis}\, 0°}}$

$= \underline{\underline{1}}$

4 a) Beh.: $\text{cis}\,\varphi \cdot \text{cis}\,(-\varphi) = 1$

Beweis: $\text{cis}\,\varphi \cdot \text{cis}\, -\varphi = \text{cis}\,(\varphi + (-\varphi))$

$= \text{cis}\, 0° = 1$ "

b) Beh.: $\text{cis}\,\varphi : \text{cis}\,(-\varphi) = \text{cis}\, 2\varphi$

Beweis: $\text{cis}\,\varphi : \text{cis}\,(-\varphi) = \text{cis}\,(\varphi - (-\varphi)) = \text{cis}\, 2\varphi$ "

⑤ a) $\cos 105° = \operatorname{Re}(\operatorname{cis} 105°)$

$= \operatorname{Re}(\operatorname{cis} 60° \cdot \operatorname{cis} 45°)$

$= \operatorname{Re}\left(\left(\frac{1}{2} + \frac{\sqrt{3}}{2} \cdot i\right) \cdot \left(\frac{\sqrt{2}}{2} + i \cdot \frac{\sqrt{2}}{2}\right)\right)$

$= \underline{\underline{\frac{\sqrt{2}}{4} - \frac{\sqrt{6}}{4}}}$

b) $\sin 15° = \operatorname{Im}\left(\frac{\operatorname{cis} 45°}{\operatorname{cis} 30°}\right)$

$= \operatorname{Im}\left(\frac{\frac{\sqrt{2}}{2} + i \cdot \frac{\sqrt{2}}{2}}{\frac{\sqrt{3}}{2} + i \cdot \frac{1}{2}}\right)$

$= \operatorname{Im}\left(\frac{\left(\frac{\sqrt{2}}{2} + i \cdot \frac{\sqrt{2}}{2}\right) \cdot \left(\frac{\sqrt{3}}{2} - i \cdot \frac{1}{2}\right)}{\frac{3}{4} + \frac{1}{4}}\right)$

$= \underline{\underline{-\frac{\sqrt{2}}{4} + \frac{\sqrt{6}}{4}}}$