Analysis - Aufgaben: Integralrechnung 2

① mit partieller Integration: $\int f'g = fg - \int fg'$

a) $\int_1^e x \cdot \ln x \, dx$

Wähle $f'(x) = x \Rightarrow f(x) = \frac{1}{2} \cdot x^2$
$g(x) = \ln x \Rightarrow g'(x) = \frac{1}{x}$

$$= \frac{1}{2} \cdot x^2 \cdot \ln x \Big|_1^e - \int_1^e \frac{1}{2} \cdot x^2 \cdot \frac{1}{x} \, dx$$

$$= \frac{1}{2} e^2 - 0 - \underbrace{\int_1^e \frac{1}{2} x \, dx}$$

$$= \frac{1}{4} \cdot x^2 \Big|_1^e$$

$$= \frac{1}{4} e^2 - \frac{1}{4}$$

$$= \frac{1}{2} e^2 - \frac{1}{4} e^2 + \frac{1}{4}$$

$$= \underline{\underline{\frac{1}{4} (e^2 + 1)}}$$

b) $\int_0^1 x e^{2x} \, dx$

Wähle $f'(x) = e^{2x} \Rightarrow f(x) = \frac{1}{2} e^{2x}$
$g(x) = x \Rightarrow g'(x) = 1$

$$= \frac{1}{2} \cdot e^{2x} \cdot x \Big|_0^1 - \int_0^1 \frac{1}{2} e^{2x} \cdot 1 \, dx$$

$$= \frac{1}{2} e^2 - 0 - \underbrace{\frac{1}{4} e^{2x} \Big|_0^1}$$

$$= \frac{1}{4} e^2 - \frac{1}{4}$$

$$= \frac{1}{2} e^2 - \frac{1}{4} e^2 + \frac{1}{4}$$

$$= \underline{\underline{\frac{1}{4} (e^2 + 1)}}$$

c) $\int_0^\pi x \cdot \cos x \, dx$

Wähle $f'(x) = \cos x \Rightarrow f(x) = \sin x$
$g(x) = x \Rightarrow g'(x) = 1$

$$= \sin x \cdot x \Big|_0^\pi - \int_0^\pi \sin x \, dx$$

$$= \sin \pi \cdot \pi - 0 - \underbrace{(-\cos x) \Big|_0^\pi}$$

$$= -\cos \pi + \cos 0$$

$$= 1 + 1$$

$$= 0 - 2$$

$$= \underline{\underline{-2}}$$

d) $\int_0^1 (1+x) e^x \, dx$

Wähle $f'(x) = e^x \Rightarrow f(x) = e^x$
$g(x) = 1 + x \Rightarrow g'(x) = 1$

$$= e^x (1+x) \Big|_0^1 - \int_0^1 e^x \, dx$$

$$= 2e - 1 - e^x \Big|_0^1$$

$$= 2e - 1 - (e - 1)$$

$$= \underline{\underline{e}}$$

e) $$\int_1^2 \frac{1}{x^2} \cdot \ln x \, dx$$

Wähle $f'(x) = x^{-2} \Rightarrow f(x) = (-1)\,x^{-1}$

$g(x) = \ln x \Rightarrow g'(x) = \frac{1}{x}$

$$= -\frac{1}{x} \cdot \ln x \,\Big|_1^2 - \int_1^2 -\frac{1}{x} \cdot \frac{1}{x} \, dx$$

$$= -\frac{1}{2} \cdot \ln 2 - 0 + \underbrace{\int_1^2 x^{-2} \, dx}$$

$$= (-1)\,x^{-1} \,\Big|_1^2$$

$$= -\frac{1}{2} + 1$$

$$= \underline{\underline{\frac{1}{2} \cdot (1 - \ln 2)}} \qquad (0{,}153)$$

f) $$\int_0^{\pi/4} \frac{x}{\cos^2 x} \, dx$$

Wähle $f'(x) = \frac{1}{\cos^2 x} \overset{FuT}{\Rightarrow} f(x) = \tan x$

$g(x) = x \Rightarrow g'(x) = 1$

$$= x \cdot \tan x \,\Big|_0^{\pi/4} - \int_0^{\pi/4} \tan x \, dx$$

$$= \frac{\pi}{4} - 0 - \underbrace{\left(-\ln|\cos x|\right)\Big|_0^{\pi/4}}$$

$$= -\ln\left|\cos \frac{\pi}{4}\right| + \ln|\cos 0|$$

$$= -\ln\left(\frac{\sqrt{2}}{2}\right) + \ln 1$$

$$= \frac{\pi}{4} + \ln\left(\frac{\sqrt{2}}{2}\right) - 0$$

$$= \underline{\underline{0{,}4388\ldots}}$$

g) $$\int_1^e (\ln x)^2 \, dx$$

Wähle $f'(x) = \ln x \overset{FuT}{\Rightarrow} f(x) = x \cdot (\ln|x| - 1)$

$g(x) = \ln x \Rightarrow g'(x) = \frac{1}{x}$

$$= \ln x \cdot x \cdot (\ln x - 1)\Big|_1^e - \int_1^e \frac{1}{x} \cdot x \cdot (\ln x - 1) \, dx$$

$$= 0 - 0 - \underbrace{\int_1^e \ln x - 1 \, dx}$$

$$= x \cdot (\ln|x| - 1) - x \,\Big|_1^e$$

$$= (0 - e) - (-1 - 1)$$

$$= \underline{\underline{e - 2}}$$

h) $$\int x \cdot \sin 2x \, dx$$

Wähle $f'(x) = \sin 2x \Rightarrow f(x) = -\frac{1}{2} \cdot \cos 2x$

$g(x) = x \Rightarrow g'(x) = 1$

$$= -\frac{1}{2} \cdot x \cdot \cos 2x + C_1 - \int -\frac{1}{2} \cdot \cos 2x \, dx$$

$$= -\frac{1}{2} x \cdot \cos 2x + C_1 + \frac{1}{2} \cdot \underbrace{\int \cos 2x \, dx}_{= \frac{1}{2} \cdot \sin 2x + C_2}$$

$$= -\frac{1}{2} x \cdot \cos 2x + C_1 + \frac{1}{4} \cdot \sin 2x + C_2$$

$$= \underline{\underline{-\frac{1}{2} x \cdot \cos 2x + \frac{1}{4} \cdot \sin 2x + C}}$$

i) $\int x\sqrt{1+2x}\, dx = \dots$

Wähle $f'(x) = (1+2x)^{1/2} \Rightarrow f(x) = \frac{1}{2} \cdot \frac{2}{3} (1+2x)^{3/2}$

$= \frac{1}{3} \cdot (1+2x)^{3/2}$

$g(x) = x \Rightarrow g'(x) = 1$

$$\dots = \frac{1}{3} \cdot x \cdot (1+2x)^{3/2} + C_1 - \underbrace{\int \frac{1}{3} \cdot (1+2x)^{3/2}\, dx}$$

$= \frac{1}{3} \cdot \int (1+2x)^{3/2}\, dx$

$= \frac{1}{3} \cdot \frac{1}{2} \cdot \frac{2}{5} \cdot (1+2x)^{5/2} + C_2$

$= \frac{1}{15} \cdot (1+2x)^{5/2} + C_2$

$$= \underline{\underline{\frac{1}{3} x (1+2x)^{3/2} - \frac{1}{15} (1+2x)^{5/2} + C}}$$

j) $\int x^2 e^{1/2 x}\, dx$

Wähle $f'(x) = e^{1/2 \cdot x} \Rightarrow f(x) = 2e^{1/2 \cdot x}$

$g(x) = x^2 \Rightarrow g'(x) = 2x$

$$= 2x^2 \cdot e^{1/2 x} + C_1 - \underbrace{\int 4x e^{1/2 \cdot x}\, dx}$$

Wähle $a'(x) = e^{1/2 \cdot x} \Rightarrow a(x) = 2 \cdot e^{1/2 x}$

$b(x) = 4x \Rightarrow b'(x) = 4$

$= 8x e^{1/2 x} + C_2 - \int 8 e^{1/2 x}\, dx$

$= 8x e^{1/2 x} + C_2 - 16 e^{1/2 \cdot x} + C_3$

$$= 2x^2 e^{1/2 x} + C_1 - 8x e^{1/2 \cdot x} - C_2 + 16 e^{1/2 \cdot x} - C_3$$

$$= \underline{\underline{(2x^2 - 8x + 16) e^{1/2 \cdot x} + C}}$$

k) $\underbrace{\int \frac{\tan x}{\cos^2 x}\,dx}$

Wähle $f'(x) = \frac{1}{\cos^2 x} \overset{FuT}{\Rightarrow} f(x) = \tan x$

$g(x) = \tan x \overset{FuT}{\Longrightarrow} g'(x) = \frac{1}{\cos^2 x}$

$\Bigg\} = \tan^2 x + C_1 - \int \frac{\tan x}{\cos^2 x}\,dx$

$$\Leftrightarrow \quad 2 \cdot \int \frac{\tan x}{\cos^2 x}\,dx = \tan^2 x + C_1$$

$$\Leftrightarrow \quad \int \frac{\tan x}{\cos^2 x}\,dx = \tfrac{1}{2} \cdot \tan^2 x + C$$

l) $\underbrace{\int \sin 2x \cdot \cos \tfrac{1}{2}x\,dx}$

Wähle $f'(x) = \sin 2x \Rightarrow f(x) = -\tfrac{1}{2} \cdot \cos 2x$

$g(x) = \cos \tfrac{1}{2}x \Rightarrow g'(x) = -\tfrac{1}{2} \cdot \sin \tfrac{1}{2}x$

$\Bigg\} = -\tfrac{1}{2} \cdot \cos 2x \cdot \cos \tfrac{1}{2}x + C_1$

$- \tfrac{1}{4} \underbrace{\int \cos 2x \cdot \sin \tfrac{1}{2}x\,dx}$

Wähle $c'(x) = \cos 2x \Rightarrow c(x) = \tfrac{1}{2} \cdot \sin 2x$

$s(x) = \sin \tfrac{1}{2}x \Rightarrow s'(x) = \tfrac{1}{2} \cdot \cos \tfrac{1}{2}x$

$= \tfrac{1}{2} \cdot \sin 2x \cdot \sin \tfrac{1}{2}x + C_2 - \tfrac{1}{4} \cdot \int \sin 2x \cdot \cos \tfrac{1}{2}x\,dx$

$= -\tfrac{1}{2} \cdot \cos 2x \cdot \cos \tfrac{1}{2}x + C_1 - \tfrac{1}{8} \cdot \sin 2x \sin \tfrac{1}{2}x - C_2 + \tfrac{1}{16} \int \sin 2x \cos \tfrac{1}{2}x\,dx$

$$\Leftrightarrow \int \sin 2x \cdot \cos \tfrac{1}{2}x\,dx = -\tfrac{1}{2} \cos 2x \cdot \cos \tfrac{1}{2}x + C_1 - \tfrac{1}{8} \sin 2x \sin \tfrac{1}{2}x - C_2 + \tfrac{1}{16} \int \sin 2x \cos \tfrac{1}{2}x\,dx$$

$$\Leftrightarrow \tfrac{15}{16} \int \sin 2x \cdot \cos \tfrac{1}{2}x\,dx = -\tfrac{1}{2} \cdot \cos 2x \cdot \cos \tfrac{1}{2}x + C_1 - \tfrac{1}{8} \cdot \sin 2x \sin \tfrac{1}{2}x - C_2$$

$$\Leftrightarrow \int \sin 2x \cdot \cos \tfrac{1}{2}x\,dx = -\tfrac{8}{15} \cos 2x \cdot \cos \tfrac{1}{2}x - \tfrac{2}{15} \sin 2x \sin \tfrac{1}{2}x + C$$

Analysis-Aufgaben: Integralrechnung 2

② mit der Substitutionsregel: $\int_0^3 f(g(x))\cdot g'(x)\,dx = \int_{g(0)}^{g(3)} f(g)\,dg$

a) $\int_0^1 (1-2x)^2\,dx$ $=$ " $\int_{g(0)}^{g(1)} g^2\cdot -\frac{1}{2}\,dg$ "

Setze $g(x) = 1-2x$

$\Rightarrow g'(x) = \frac{dg}{dx} = -2$

$\Leftrightarrow dx = -\frac{1}{2}\cdot dg$

$= -\frac{1}{2}\cdot\int_1^{-1} g^2\,dg$

$= -\frac{1}{2}\cdot \frac{1}{3}g^3\Big|_1^{-1}$

$= -\frac{1}{2}\cdot\left(-\frac{1}{3}-\frac{1}{3}\right)$

$= \underline{\underline{\frac{1}{3}}}$

(Kontrolle: $(1-2x)^2 = 1-4x+4x^2$

$\Rightarrow \underline{\int_0^1 (1-2x)^2\,dx} = \int_0^1 (1-4x+4x^2)\,dx$

$= x-2x^2+\frac{4}{3}x^3\Big|_0^1$

$= 1-2+\frac{4}{3} - 0 \quad = \underline{\frac{1}{3}}$ ✓)

b) $\int_0^2 2\cdot\left(\frac{1}{2}-t\right)^3\,dt$ $=$ " $\int_{g(0)}^{g(2)} 2\cdot g^3\cdot(-1)\,dg$ "

Setze $g(t) = \frac{1}{2}-t$

$\Rightarrow g'(t) = \frac{dg}{dt} = -1$

$\Leftrightarrow dt = -dg$

$= (-2)\cdot\int_{1/2}^{-3/2} g^3\,dg$

$= (-2)\cdot \frac{1}{4}g^4\Big|_{1/2}^{-3/2}$

$= (-2)\cdot\left(\frac{1}{4}\cdot\frac{81}{16} - \frac{1}{4}\cdot\frac{1}{16}\right)$

$= (-2)\cdot\frac{80}{64}$

$= \underline{\underline{-2{,}5}}$

c) $\int_0^4 \sqrt{4-r}\,dr$ $=$ " $\int_{g(0)}^{g(4)} \sqrt{g}\cdot(-1)\,dg$ "

Setze $g(r) = 4-r$

$\Rightarrow g'(r) = \frac{dg}{dr} = -1$

$\Leftrightarrow dr = -dg$

$= (-1)\cdot\int_4^0 g^{1/2}\,dg$

$= (-1)\cdot \frac{2}{3}g^{3/2}\Big|_4^0$

$= (-1)\cdot\left(0-\frac{2}{3}\cdot\underbrace{\sqrt[2]{4^3}}_{8}\right)$

$= \underline{\underline{5\frac{1}{3}}}$

d) $\int_a^3 \sqrt{px+q}\,dx \;=\; "\int_{g(a)}^{g(3)} \sqrt{g}\cdot 1/p\,dg"$

Setze $g(x) = px+q$

$\Rightarrow g'(x) = \frac{dg}{dx} = p$

$\Leftrightarrow dx = 1/p \cdot dg$

$= 1/p \cdot \int_{pa+q}^{p3+q} g^{1/2}\,dg$

$= 1/p \cdot \; 2/3\, g^{3/2} \Big|_{pa+q}^{p3+q}$

$= \frac{2}{3p} \cdot \left((p3+q)^{3/2} - (pa+q)^{3/2} \right)$

e) $\int_{-1}^{1} 3\cdot e^{-4x}\,dx \;=\; "\int_{g(-1)}^{g(1)} 3\cdot e^{g}\cdot(-1/4)\,dg"$

Setze $g(x) = -4x$

$\Rightarrow g'(x) = \frac{dg}{dx} = -4$

$\Leftrightarrow dx = -1/4\,dg$

$= -3/4 \int_{4}^{-4} e^{g}\,dg$

$= -3/4 \cdot \; e^{g}\Big|_{4}^{-4}$

$= -3/4 \cdot (e^{-4} - e^{4})$

$= 40{,}93$

f) $\int_{1}^{\ln 2} \sqrt{e^{-x}}\,dx \;=\; "\int_{g(1)}^{g(\ln 2)} \sqrt{e^{g}}\cdot(-1)\,dg"$

Setze $g(x) = -x$

$\Rightarrow g'(x) = \frac{dg}{dx} = -1$

$\Leftrightarrow dx = -dg$

$= (-1)\cdot \int_{-1}^{-\ln 2} e^{g/2}\,dg$

$= (-1) \cdot \; 2\cdot e^{g/2}\Big|_{-1}^{-\ln 2}$

$= (-2)\cdot \left(e^{-1/2\cdot \ln 2} - e^{-1/2}\right)$

$= (-2)\cdot \left(2^{-1/2} - e^{-1/2}\right)$

$= -0{,}201$

g) $\int_a^3 \cos(cx+d)\,dx \;=\; "\int_{g(a)}^{g(3)} \cos(g)\cdot 1/c\,dg"$

Setze $g(x) = cx+d$

$\Rightarrow g'(x) = \frac{dg}{dx} = c$

$\Leftrightarrow dx = 1/c \cdot dg$

$= 1/c \cdot \int_{ca+d}^{c3+d} \cos g\,dg$

$= 1/c \cdot \; (+\sin g)\Big|_{ca+d}^{c3+d}$

$= +1/c \cdot \left(\sin(c3+d) - \sin(ca+d)\right)$

h) $\int \frac{1+x}{1-x^2}\,dx = \int \frac{1}{1-x}\,dx = {}$ " $\int \frac{1}{g} \cdot (-1)\,dg$ "

Setze $g(x) = 1-x$ $\qquad = (-1) \cdot \int g^{-1}\,dg$

$\Rightarrow g'(x) = \frac{dg}{dx} = -1$ $\qquad = (-1) \cdot \ln|g| + C$

$\Leftrightarrow dx = -dg$ $\qquad \overset{\text{Rück-}}{\underset{\text{subst.}}{=}} \underline{\underline{(-1) \cdot \ln|1-x| + C}}$ ✓. (vgl FoT)

i) $\int 3x \sqrt[3]{4-2x^2}\,dx = {}$ " $\int 3x \cdot \sqrt[3]{g} \cdot \frac{dg}{-4x}$ "

Setze $g(x) = 4-2x^2$ $\qquad = -\frac{3}{4} \cdot \int g^{1/3}\,dg$

$\Rightarrow g'(x) = \frac{dg}{dx} = -4x$ $\qquad = -\frac{3}{4} \cdot \frac{3}{4}\, g^{4/3} + C$

$\Leftrightarrow dx = \frac{dg}{-4x}$ $\qquad \overset{\text{Rück-}}{\underset{\text{subst.}}{=}} \underline{\underline{-\frac{9}{16} \cdot (4-2x^2)^{4/3} + C}}$ (Kontrolle: durch Ableiten!)

j) $\int x^{-2} e^{1/x}\,dx = {}$ " $\int x^{-2} \cdot e^{g} \cdot (-1)\,dg \cdot x^2$ "

Setze $g(x) = 1/x = x^{-1}$ $\qquad = (-1) \int e^{g}\,dg$

$\Rightarrow g'(x) = \frac{dg}{dx} = -x^{-2}$ $\qquad = (-1) \cdot e^{g} + C$

$\Leftrightarrow dx = -dg \cdot x^2$ $\qquad = \underline{\underline{(-1)\, e^{1/x} + C}}$

k) $\int x \cdot \sin x^2\,dx = {}$ " $\int x \cdot \sin g \cdot \frac{dg}{2x}$ "

Setze $g(x) = x^2$ $\qquad = \frac{1}{2} \cdot \int \sin g\,dg$

$\Rightarrow g'(x) = \frac{dg}{dx} = 2x$ $\qquad = \frac{1}{2} \cdot (-\cos g) + C$

$\Leftrightarrow dx = \frac{dg}{2x}$ $\qquad = \underline{\underline{-\frac{1}{2} \cdot \cos x^2 + C}}$

l) $\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx = {}$ " $\int \frac{\sin x}{g} \cdot \frac{dg}{-\sin x}$ "

Setze $g(x) = \cos x$ $\qquad = -\int g^{-1}\,dg$

$\Rightarrow g'(x) = \frac{dg}{dx} = -\sin x$ $\qquad = -\ln|g| + C$

$\Leftrightarrow dx = \frac{dg}{-\sin x}$ $\qquad = \underline{\underline{-\ln|\cos x| + C}}$ ✓. (vgl FoT)

Analysis-Aufg.: Int.rechnung 2

Subst.regel: $\int_a^b f(g(x)) \cdot g'(x)\,dx = \int_{g(a)}^{g(b)} f(g)\,dg$

③ Lösen mit vorgegebener Substitution:

a) $\int_1^2 \sqrt{e^{3x}+e^{2x}}\,dx$, $x(t) = \ln t \iff t = e^{x(t)}$

$\Rightarrow \frac{dx}{dt} = \frac{1}{t} \iff dx = \frac{dt}{t}$

$$\int_{e^1}^{e^2} \sqrt{e^{3\cdot\ln t}+e^{2\cdot\ln t}}\cdot\frac{dt}{t} = \int_e^{e^2} \sqrt{t^3+t^2}\,\frac{dt}{t}$$

$$= \int_e^{e^2} t\cdot\sqrt{t+1}\,\frac{dt}{t}$$

$$= \int_e^{e^2} \sqrt{t+1}\,dt$$

neue Subst.: $g(t) := t+1$

$\iff \frac{dg}{dt} = 1 \iff dt = dg$

$$\Rightarrow \int_{e+1}^{e^2+1} g^{1/2}\,dg$$

$$= \frac{2}{3}\, g^{3/2} \Big|_{e+1}^{e^2+1} = \ldots = \underline{\underline{11,419}}$$

5) $\displaystyle\int_0^{\frac{1}{2}\cdot\sqrt{2}} \frac{1}{\sqrt{1-x^2}}\,dx$, $x(t) = \sin t \iff t = \sin^{-1}(x)$

$\iff \frac{dx}{dt} = \cos t \iff dx = \cos t \cdot dt$.

Subst.

$$\int_{\sin^{-1}(0)}^{\sin^{-1}\left(\frac{\sqrt{2}}{2}\right)} \frac{1}{\sqrt{1-\sin^2 t}} \cdot \cos t \, dt = \int_0^{\pi/4} 1 \, dt = \underline{\underline{\pi/4}}$$

(4) "eine" Stammfunktion $\Rightarrow$ Setze jeweils $C = 0$.

a) $A(x) = -\frac{3}{45} x^9$

b) $H(u) = \frac{1}{15} \cdot u^{15} - \frac{30}{11} \cdot u^{11} + \frac{45}{7} \cdot u^7$

c) $R(s) = \frac{3t^4 - 3t^2 + 5t - 7}{4t^2} \cdot s$

d) $G(x) = \frac{1}{2} \cdot e^{2x} \left(x^2 - x + \frac{1}{2}\right)$

e) $R(a) = \frac{1}{2} \cdot s \cdot \sin^2 a$

f) $G(x) = \frac{1}{21} \cdot (3x-5)^7$

g) $H(t) = \frac{1}{3} \cdot (2t-1)^{3/2}$

h) $F(t) = \frac{1}{(n+1)!} t^{n+1}$

i) $R(x) = e^{\sin x}$

j) $G(x) = \frac{1}{2} (e^x - e^{-x})$

k) $K(z) \overset{*}{=} \frac{1}{4} \cdot \sin z \cdot \cos^3 z - \frac{3}{8} \cdot \sin z \cdot \cos z + \frac{3}{8} z$

(Verwende: $\sin^2 x + \cos^2 x = 1$, $\sin 2x = 2 \sin x \cdot \cos x$, $\cos 2x = 2\cos^2 x - 1$)

e) $B(t) = -t \cdot e^{-t} - e^{-t}$

m) $C(x) = \frac{1}{2} x^2 \cdot (\ln x - \frac{1}{2})$

n) $F(s) = -\frac{\ln s}{s} - \frac{1}{s}$

o) $J(r) = -\frac{1}{2} \cdot \cos^2 r$

(5) a) $-\frac{26}{27}$

b) $\frac{1}{4}$

c) $\frac{2^{-3/2}}{3}$

4k) $\int \cos^4 z \, dz$

$\int \cos^4 z \, dz = \int (1-\sin^2 z)\cdot \cos^2 z \, dz$ $\quad (\sin^2 + \cos^2 = 1)$

$= \int \cos^2 z \, dz - \underbrace{\int \sin^2 z \cdot \cos^2 z \, dz}$

$= \int (\sin z \cdot \cos z)^2 \, dz$

$= \int \frac{1}{4} \cdot \sin^2(2z) \, dz$ $\quad (\sin 2x = 2\sin x \cdot \cos x)$

$= \int \cos^2 z \, dz - \frac{1}{4} \int \sin^2 2z \, dz$

- $\int \cos^2 z \, dz = \int \cos z \cdot \cos z \, dz$ $= \cos z \cdot \sin z + C_1 - \int -\sin^2 z \, dz$

$f(z) = \cos z \Rightarrow f'(z) = -\sin z$

$g'(z) = \cos z \Rightarrow g(z) = \sin z$

$= \cos z \cdot \sin z + C_1 + \int \sin^2 z \, dz$

$= \cos z \cdot \sin z + C_1 + \int 1 - \cos^2 z \, dz$

$= \cos z \cdot \sin z + C_1 + \int 1 \, dz - \int \cos^2 z \, dz$

$\Leftrightarrow 2 \cdot \int \cos^2 z \, dz = \cos z \cdot \sin z + C_1 + z + C_2$

$\Leftrightarrow \int \cos^2 z \, dz = \frac{\sin z \cdot \cos z + z}{2} + C$

- $\int \sin^2 z \, dz = \int 1 - \cos^2 z \, dz$

$= \int 1 \, dz - \int \cos^2 z \, dz$

$= z - \left(\frac{\sin z \cdot \cos z + z}{2} + C\right) \overset{(*)}{=} \frac{z - \sin z \cdot \cos z}{2} + \tilde{C}$

- $\int \sin^2 2z \, dz$ $\quad u(z) = 2z \;=\; \frac{du}{dz} = 2$

$\int \sin^2 u \cdot \frac{1}{2} \, du = \frac{1}{2} \cdot \int \sin^2 u \, du$

$\overset{(*)}{=} \frac{1}{2} \cdot \left(\frac{u - \sin u \cdot \cos u}{2} + C\right)$

$\overset{\text{Rück-subst}}{=} \frac{2z - \sin 2z \cdot \cos 2z}{4} + \tilde{C}$

$$\Rightarrow \int \cos^4 z\,dz = \frac{\sin z \cdot \cos z + z}{2} + C - \frac{1}{4} \cdot \frac{2z - \sin 2z \cdot \cos 2z}{4} + \hat{C}$$

$$= \frac{\sin z \cdot \cos z}{2} + \frac{z}{2} - \frac{z}{8} + \frac{\sin 2z \cdot \cos 2z}{16} + \hat{C}$$

$\sin 2z = 2 \cdot \sin z \cdot \cos z$

$\cos 2z = 2\cos^2 z - 1$

$$= \frac{\sin z \cdot \cos z}{2} + \frac{3z}{8} + \frac{2 \cdot \sin z \cdot \cos z \cdot (2\cos^2 z - 1)}{16} + \hat{C}$$

$$= \frac{4\sin z \cdot \cos z + 3z + 2\sin z \cdot \cos^3 z - \sin z \cdot \cos z}{8} + \hat{C}$$

$$= \frac{1}{4} \cdot \sin z \cdot \cos^3 z + \frac{3}{8} \cdot \sin z \cdot \cos z + \frac{3}{8} \cdot z + \hat{C}$$