Analysis-Aufg.: Integralrechnung 5

① (Vol. formel für rot. symm. Körper (bzgl. x-Achse): $\pi \int_0^b f(x)^2\,dx$)

$f(x) = (x-2)^2 \cdot \sqrt{3x}$

NS: $x_1 = 0$, $x_2 = 2$

$f(x)^2 = (x-2)^4 \cdot 3x$

1 1
1 2 1
1 3 3 1
1 4 6 4 1

$= (x^4 - 8x^3 + 24x^2 - 32x + 16) \cdot 3x$

$= 3x^5 - 24x^4 + 72x^3 - 96x^2 + 48x$

$$\Rightarrow Vol = \pi \int_0^2 3x^5 - 24x^4 + 72x^3 - 96x^2 + 48x \, dx$$

$$= \pi \cdot \left. \tfrac{1}{2} \cdot x^6 - \tfrac{24}{5} \cdot x^5 + 18x^4 - 32x^3 + 24x^2 \right|_0^2$$

$$= \pi \cdot [(32 - 153{,}6 + 288 - 256 + 96) - (0)]$$

$$= \underline{6.4 \cdot \pi} \quad (= 20{,}106)$$

② Lösung => Verwende Volumen $V = 1$

$f(x) = \frac{\sqrt{6}}{2} \cdot x^2 \qquad (= y \iff x^2 = \frac{2y}{\sqrt{6}})$

Rotation um y-Achse => Wir benötigen die Querschnittsfläche in Abhängigkeit von y:

$$A = A(y) = \pi \cdot \frac{2y}{\sqrt{6}}$$

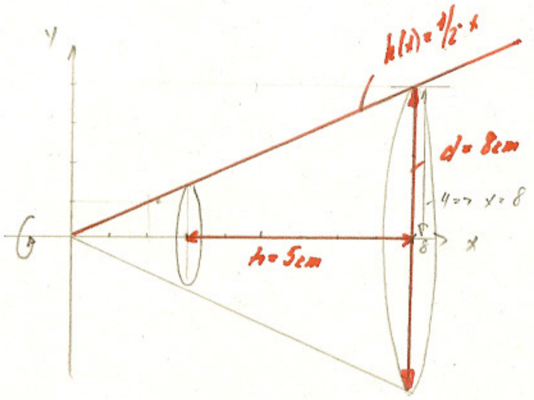
$$\Rightarrow V = \pi \int_0^h \frac{2y}{\sqrt{6}}\,dy \overset{!}{=} 1$$

$$\iff \pi \cdot \left. \frac{1}{\sqrt{6}} y^2 \right|_0^h = 1$$

$$\iff \frac{\pi \cdot h^2}{\sqrt{6}} = 1$$

$$\iff \underline{h} = \sqrt{\sqrt{6}/\pi} = \underline{0{,}883}$$

③ Situation:



$$\Rightarrow V = \pi \cdot \int_3^8 h(x)^2\,dx$$

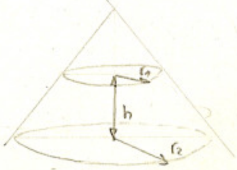
$$= \pi \cdot \int_3^8 \tfrac{1}{4} \cdot x^2\,dx$$

$$= \pi \cdot \left. \tfrac{1}{12} x^3 \right|_3^8$$

$$= \pi \cdot \left(\tfrac{1}{12} \cdot 8^3 - \tfrac{1}{12} \cdot 3^3 \right) = \ldots = \underline{126{,}57}$$

Formeln & Tafeln

$V = \frac{\pi \cdot h}{3} (r_1^2 + r_2^2 + r_1 \cdot r_2)$



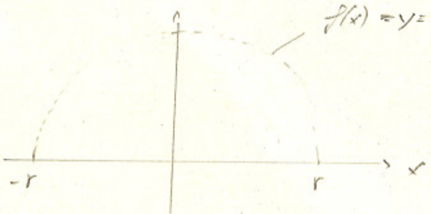
$h = 5$
$r_1 = h(3) = 1{,}5$
$r_2 = h(8) = 4$

$\Rightarrow \underline{V = \ldots = 126{,}57}$ ✓

4) Beh.: $V_{Kugel} = \frac{4\pi}{3} \cdot r^3$

Beweis.

$f(x) = y = \sqrt{r^2 - x^2}$



$$\Rightarrow \underline{V} = \pi \cdot \int_{-r}^{r} f^2(x)\, dx$$

$$= \pi \int_{-r}^{r} r^2 - x^2 \, dx$$

$$= \pi \cdot \left[x \cdot r^2 - \tfrac{1}{3} \cdot x^3 \right]_{-r}^{r}$$

$$= \pi \cdot \left[\underbrace{\left(r^3 - \tfrac{1}{3} r^3\right)}_{\frac{2}{3} \cdot r^3} - \underbrace{\left(-r^3 + \tfrac{1}{3} r^3\right)}_{-\frac{2}{3} \cdot r^3} \right] = \underline{\underline{\frac{4\pi}{3} r^3}}$$

□

⑤ a) Polynomfkt. 4. Ordnung $\Rightarrow f(x) = ax^4 + bx^3 + cx^2 + dx + e$

hat in (0|0) einen WP $\Rightarrow f(0) = 0$, $f''(0) = 0$

W'tangente = x-Achse $\Rightarrow f'(0) = 0$

$A \in$ graph(f) $\Rightarrow f(-4) = 0$

$B \in$ graph(f) $\Rightarrow f(2) = 2$

$\Rightarrow a = 1/24 = 0{,}042$, $b = 1/6 = 0{,}167$, $c = d = e = 0$

$$\Rightarrow \underline{\underline{f(x) = \frac{x^4}{24} + \frac{x^3}{6}}}$$

b) Für die Kurvendiskussion.

· NS: $f(x) = 0 \iff x^3\left(\frac{1}{24}\cdot x + \frac{1}{6}\right) = 0$

$\Rightarrow \underline{\underline{x_1 = x_2 = x_3 = 0}}, \quad \underline{\underline{x_4 = -4}}$

· lokale Extrema: $f'(x) = \frac{1}{6}\cdot x^3 + \frac{1}{2}x^2 \overset{!}{=} 0$

$\iff x^2\left(\frac{1}{6}\cdot x + \frac{1}{2}\right) = 0$

$\iff \underline{\underbrace{x_5 = x_6 = 0}_{\text{sind WP}}, \; x_7 = -3}$

$f''(x) = \frac{1}{2}\cdot x^2 + x \Rightarrow f''(-3) = 1{,}5 > 0 \Rightarrow$ lokales Minimum an der Stelle $\underline{x_7 = -3}$ mit Wert $\underline{\underline{y = -\frac{9}{8}}}$

· WP: $f''(x) = \frac{1}{2}\cdot x^2 + x = x\left(\frac{1}{2}x + 1\right) \overset{!}{=} 0$

$\iff \underline{\underline{x = 0, \; y = 0}}$

$\underline{\underline{x = -2, \; y = -\frac{2}{3}}}$

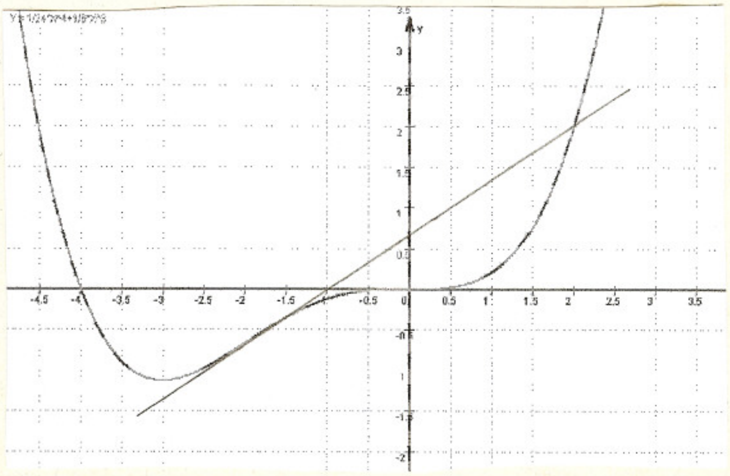
c) i) Funktionsgleichung der Wendetangente ≠ x-Achse:

$t = t(x) = cx + b$

mit · $\underline{c} = f'(-2) = \underline{\frac{2}{3}}$

· $\left(-2 \,|\, -\frac{2}{3}\right) \in \text{graph}(t) \iff t(-2) = -\frac{2}{3}$

$\Rightarrow \underline{b = \frac{2}{3}}$



ii) Fläche zwischen graph(t) & graph(f)

$\underline{\underline{A}} = \int_{-2}^{2} t(x) - f(x)\, dx = \ldots = \underline{\underline{2\frac{2}{15}}}$

⑥ $V_f = \pi \cdot \int_c^3 f(x)^2 dx$

$V_g = \pi \int_c^3 g(x)^2 dx$

$\Rightarrow \underline{\underline{V}} = V_g - V_f$

$= \pi \int_c^3 g(x)^2 dx - \pi \int_c^3 f(x)^2 dx = \underline{\underline{\pi \cdot \int_c^3 g(x)^2 - f(x)^2 dx}}$

⑦ $f(x) = 1 - \frac{2e^x}{e^x + c} = 0$

$\Leftrightarrow \quad 2e^x = e^x + c$

$\Leftrightarrow \quad e^x = c$

$\Leftrightarrow \quad \underline{x = \ln c}$ = einzige NS(f)

$\Rightarrow$ für die Fläche folgt: $A = \int_{\ln c}^{\ln(2c)} f(x)\, dx$

$= \int_{\ln c}^{\ln(2c)} 1\, dx - \underbrace{\int_{\ln c}^{\ln(2c)} \frac{2e^x}{e^x + c}\, dx}$

$\Rightarrow$ Sei $g(x) := e^x + c$

$\Rightarrow \frac{dg}{dx} = e^x \Leftrightarrow dx = \frac{dg}{e^x}$

$= x \Big|_{\ln(c)}^{\ln(2c)} - \text{"}\int_{g(\ln c)}^{g(\ln 2c)} \frac{2e^x}{g} \cdot \frac{dg}{e^x}\text{"}$

$= \ln 2c - \ln c - \int_{c+c}^{2c+c} \frac{2}{g}\, dg$

$= \ln\left(\frac{2c}{c}\right) - 2 \cdot \left(\ln(3c) - \ln(2c)\right)$

$= \ln 2 - 2 \cdot \ln \frac{3}{2}$

$= \ln 2 - \ln\left(\frac{3}{2}\right)^2$

$= \ln \frac{2}{\frac{9}{4}}$

$= \underline{\underline{\ln \frac{8}{9}}}$

⑧ $f_c(x) = c \cdot \sin x \cdot \cos x \quad , \quad c \in \mathbb{R}_{>0} \quad , \quad 0 \le x \le 2\pi$

a) NS: $f_c(x) = 0 \iff \begin{cases} \sin x = 0 & \iff \underline{\underline{x_1 = 0 ,\ x_2 = \pi ,\ x_3 = 2\pi}} \\ \text{oder} & \\ \cos x = 0 & \iff \underline{\underline{x_4 = \pi/2 \ ,\ x_5 = \frac{3\pi}{2}}} \end{cases}$

ES: $\underline{f_c'(x)} = c \cdot (\cos x \cdot \cos x + \sin x \cdot (-\sin x))$

$= \underline{c \cdot (\cos^2 x - \sin^2 x)} \qquad \left(\overset{\text{Add.}}{\underset{\text{Theorem}}{=}} \quad c \cdot \cos 2x \right)$

$\Rightarrow f_c'(x) = 0 \iff \cos^2 x = \sin^2 x$

$\iff \begin{cases} \cos x = \sin x & \iff \underline{x_6 = \pi/4 \ ,\ x_7 = 5\pi/4} \\ \text{oder} & \\ \cos x = -\sin x & \iff \underline{x_8 = 3\pi/4 \ ,\ x_9 = 7\pi/4} \end{cases}$

WS: $\underline{f_c''(x)} = c \cdot (2 \cdot (-\sin x \cos x) - 2 \cdot \cos x \cdot \sin x)$

$= \underline{-4c \cdot \sin x \cdot \cos x}$

$\Rightarrow f_c'' = 0 \iff \begin{cases} \sin x = 0 & \iff \underline{x_{10} = x_1 = 0 ,\ x_{11} = x_2 = \pi ,\ x_{12} = x_3 = 2\pi} \\ \text{od.} & \\ \cos x = 0 & \iff \underline{x_{13} = x_4 = \pi/2 \ ,\ x_{14} = x_5 = \frac{3\pi}{2}} \end{cases}$

$\Rightarrow \underline{\underline{x_{6,7,8,9} \text{ sind ES}}}$

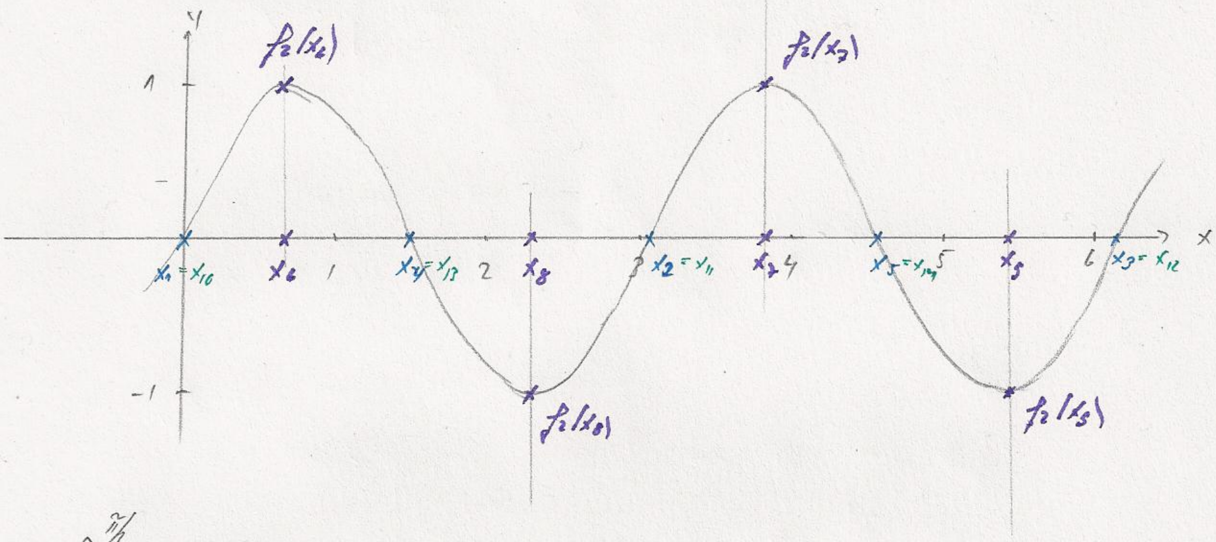
$f_c'''(x) = -4c \cdot (\cos x \cdot \cos x + \sin x \cdot (-\sin x))$

$= -4c\,(\cos^2 x - \sin^2 x) \overset{!}{=} 0 \iff x_{15,\dots,18} = x_{6,\dots,9}$

$\Rightarrow \underline{\underline{x_{10,11,12,13,14} \text{ sind WS}}}$

$(\underline{WS = NS})$

b) $f_2(x) = 2 \cdot \sin x \cdot \cos x$



c) $A = 4 \cdot \int_0^{\pi/2} 2 \cdot \sin x \cdot \cos x \, dx$

$= 8 \cdot \int_0^{\pi/2} \sin x \cdot \cos x \, dx$

$\overset{\nearrow}{=} 8 \cdot \left[-\frac{1}{4} \cdot (\cos^2 x - \sin^2 x) \Big|_0^{\pi/2} \right] = \dots = \underline{\underline{4}}$

folgt aus der Bestimmung der WS