Analysis-Aufgaben: Integralrechnung 3

① AWP: $(1+e^t)\,\dot{x} = x \quad , \quad x(0) = 2$

Beh.: $x(t) = 4\cdot e^t \cdot (1+e^t)^{-1}$ ist Lösg. d. AWP

Beweis: i)
$$\underline{x(0)} = 4\cdot e^0 \cdot (1+e^0)^{-1} = 4\cdot 1 \cdot (1+1)^{-1} = \frac{4}{2} = \underline{2} \quad \checkmark$$

ii)
$$\dot{x}(t) \overset{PR}{=} 4e^t\cdot(1+e^t)^{-1} + 4e^t \cdot \underbrace{e^t}_{\text{innere Ableitung}} \cdot \underbrace{\left((-1)\cdot(1+e^t)^{-2}\right)}_{\text{äußere Ableitung}}{}_{KR}$$
$$= 4e^t\left[(1+e^t)^{-1} - e^t\cdot(1+e^t)^{-2}\right]$$
$$\Rightarrow \underline{(1+e^t)\cdot\dot{x}} = (1+e^t)\cdot 4e^t\cdot\left[(1+e^t)^{-1} - e^t(1+e^t)^{-2}\right]$$
$$= 4e^t\left[1 - e^t\cdot(1+e^t)^{-1}\right]$$
$$= 4e^t\cdot\left[\frac{(1+e^t)-e^t}{1+e^t}\right]$$
$$= 4e^t\cdot\frac{1}{1+e^t} = 4e^t\cdot(1+e^t)^{-1} = \underline{x(t)} \quad \checkmark$$

$\square$

② a)
$$\dot{x} - 3t = 0 \iff \frac{dx}{dt} - 3t = 0$$
$$\iff dx = 3t\cdot dt$$
$$\Rightarrow \int dx = \int 3t\,dt$$
$$\iff x = \tfrac{3}{2}\cdot t^2 + C$$
$$\Rightarrow \underline{\underline{x(t) = \tfrac{3}{2}\cdot t^2 + C}}$$

b)
$$\ddot{x} = t \Rightarrow \dot{x} = \int t\,dt = \tfrac{1}{2}t^2 + C_1$$
$$\Rightarrow \underline{\underline{x}} = \int \tfrac{1}{2}\cdot t^2 + C_1\,dt = \underline{\underline{\tfrac{1}{6}t^3 + C_1 t + C_2}}$$

c)
$$\dot{x} = x\cdot t \iff \frac{dx}{dt} = x\cdot t$$
$$\iff \frac{dx}{x} = t\cdot dt$$
$$\Rightarrow \int \frac{1}{x}\,dx = \int t\,dt$$
$$\iff \ln|x| = \tfrac{1}{2}t^2 + C \Rightarrow \underline{x(t)} = e^{\frac{1}{2}t^2 + C} = \underline{\tilde{C}\cdot e^{\frac{1}{2}\cdot t^2}}$$

d) $3t^2 + at - 5\dot{x} = 0 \iff 5\dot{x} = 3t^2 + at$

$\iff 5 \cdot \frac{dx}{dt} = 3t^2 + at$

$\iff 5\,dx = (3t^2 + at)\,dt$

$\Rightarrow \int 5\,dx = \int 3t^2 + at\ dt$

$\iff 5x = t^3 + a/2 \cdot t^2 + C$

$\iff \underline{\underline{x(t) = 1/5 \cdot (t^3 + a/2\, t^2) + \hat{C}}}$

e) $\dot{x}\,(1+t^2) = tx \iff \frac{dx}{dt}\,(1+t^2) = tx$

$\iff \frac{dx}{x} = \frac{t}{(1+t^2)}\,dt$

$\Rightarrow \int \frac{1}{x}\,dx = \int \frac{t}{1+t^2}\,dt$

<u>Subst.</u>: $u(t) := 1+t^2 \Rightarrow \frac{du}{dt} = 2t$

$\iff dt = \frac{du}{2t}$

$= " \int \frac{t}{u} \cdot \frac{du}{2t} "$

$= 1/2 \cdot \int \frac{1}{u}\,du$

$= 1/2 \cdot \ln|u| + C$

$\iff \ln|x| = 1/2 \cdot \ln(1+t^2) + C$

$\iff \underline{\underline{x(t)}} = e^{1/2 \cdot \ln(1+t^2) + C}$

$= \left(e^{\ln(1+t^2)}\right)^{1/2} \cdot e^{C}$

$= \underline{\underline{\tilde{C} \cdot (1+t^2)^{1/2}}}$

f) $\dot{x} = (1-x)^2 \iff \frac{dx}{dt} = (1-x)^2$

$$\iff \frac{dx}{(1-x)^2} = dt$$

$$\Rightarrow \int \frac{1}{(1-x)^2}\,dx = \int dt$$

Subst.: $u(x) = 1-x$

$\Rightarrow \frac{du}{dx} = -1$

$\iff dx = -du$

$$\int \frac{1}{u^2} \cdot (-1)\,du$$

$$= (-1) \cdot \int u^{-2}\,du$$

$$= (-1) \cdot (-1)\,u^{-1} + C$$

$$= (1-x)^{-1} + C$$

$$\iff \frac{1}{1-x} = t + \tilde{C}$$

$$\iff 1-x = \frac{1}{t+\tilde{C}}$$

$$\iff x(t) = 1 - \frac{1}{t+\tilde{C}} = \frac{t+\tilde{C}-1}{t+\tilde{C}}$$

③ a) AWP : $x\dot{x} + 1 = t$, $x(-1) = 2$

i) allg. Lösg. : $x \cdot \frac{dx}{dt} + 1 = t$

$\Leftrightarrow \quad x\,dx = (t-1)\,dt$

$\Rightarrow \quad \int x\,dx = \int t-1\,dt$

$\Leftrightarrow \quad \frac{1}{2}x^2 = \frac{1}{2}t^2 - t + C_1$

$\Leftrightarrow \quad \underline{x(t) = (t^2 - 2t + C_2)^{1/2}}$

ii) Bestimmung der Konstanten : $x(-1) = (1 + 2 + C_2)^{1/2} = 2$

$\Rightarrow \quad 1 + 2 + C_2 = 4$

$\Leftrightarrow \quad \underline{C_2 = 1}$

$\Rightarrow \quad \underline{\underline{x(t)}} = (t^2 - 2t + 1)^{1/2}$

$= \left((t-1)^2\right)^{1/2} = \underline{\underline{|t-1|}}$

b) AWP : $\dot{x} + (\cos t) \cdot x = 0$, $x(\pi/2) = 2\pi$

i) allg. Lösg. : $\frac{dx}{dt} + (\cos t) \cdot x = 0$

$\Leftrightarrow \quad \frac{dx}{x} = (-\cos t)\,dt$

$\Rightarrow \quad \int \frac{1}{x}\,dx = -\int \cos t\,dt$

$\Leftrightarrow \quad \ln|x| = -\sin t + C$

$\Rightarrow \quad \underline{\underline{x(t)}} = e^{-\sin t + C} = \underline{\underline{\tilde{C} \cdot e^{-\sin t}}}$

ii) Bestimmung der Konstanten : $x(\pi/2) = \tilde{C} \cdot e^{-\sin(\pi/2)} \overset{!}{=} 2\pi$

$\Leftrightarrow \quad \tilde{C} \cdot e^{-1} = 2\pi$

$\Leftrightarrow \quad \underline{\tilde{C} = e \cdot 2\pi}$

$\Rightarrow \quad \underline{\underline{x(t) = 2\pi \cdot e \cdot e^{-\sin t} = 2\pi \cdot e^{1-\sin t}}}$

④ a) $\dot{x} = 1 + 2 \cdot \left(\frac{x}{t}\right)$

Subst.: $u = \frac{x}{t} \iff x = u \cdot t$

$\iff \dot{x} = \dot{u}t + u$

$\overset{\text{subst.}}{\Rightarrow} \quad \dot{u}t + u = 1 + 2u$

$\iff \quad \dot{u}t = 1 + u$

$\iff \quad \frac{du}{dt} t = 1 + u$

$\iff \quad \frac{du}{1+u} = \frac{dt}{t}$

$\Rightarrow \quad \int \frac{1}{1+u}\, du = \int \frac{1}{t}\, dt$

Subst.: $g(u) = 1 + u$
$\Rightarrow \frac{dg}{du} = 1 \iff dg = du$

$= \int \frac{1}{g}\, dg$

$= \ln|g| + C_1$

$\iff \quad \ln|1+u| = \ln|t| + C_2$

$\Rightarrow \quad \underline{u+1} = e^{\ln|t| + C_2} = \underline{C_3 \cdot t}$

Rück-subst. $\Rightarrow \quad \frac{x}{t} + 1 = C_3 \cdot t$

$\iff \quad \underline{\underline{x(t) = C_3 t^2 - t}}$

5) $\dot{x} = (t+x+1)^2$

Subst.: $u = t+x+1 \iff x = u-t-1$

$\iff \dot{x} = \dot{u} - 1$

$\overset{\text{einst.}}{\Longrightarrow} \quad \dot{u} - 1 = u^2$

$\iff \dfrac{du}{dt} = u^2+1$

$\iff \dfrac{du}{u^2+1} = dt$

$\Longrightarrow \displaystyle\int \frac{1}{u^2+1}\,du = \int dt$

$\overset{\text{Int}}{\iff} \arctan u = t + C$

$\overset{\text{Rücksubst.}}{\Longrightarrow} \quad t+x+1 = \tan(t+C)$

$\iff \underline{\underline{x(t) = \tan(t+C) - t - 1}}$